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Unified $(q; \alpha, \beta, \gamma; \nu)$-deformation of one-parametric q-deformed oscillator algebras

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Abstract
We define a generalized $(q; \alpha, \beta, \gamma; \nu)$-deformed oscillator algebra and find its structure function of deformation. This algebra includes many other deformations as special cases. We give the classification of irreducible representations of the algebra. We extract the deformed oscillator with discrete spectrum of its 'quantized coordinate'. We find the eigenvalues of this operator and show that the corresponding eigenfunctions are expressed in terms of the q-deformed (generalized) Hermitian I polynomials. The asymptotic behavior of the energy levels, dependent on the deformation parameters of the oscillator, is determined.

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1. Introduction

The investigation of one-parameter deformed oscillator algebras in theoretical physics originated from the study of the dual resonance models of strong interactions [1]. The q-deformed analog of the harmonic oscillator was introduced in the well-known papers [2, 3].

In quantum field theory situations often arise where a change of the canonical commutation relations is required. The physical motivation of the study of deformed boson and fermion quanta is connected with the hope that the deformed oscillators in nonlinear systems will play the same role as the usual oscillator in the standard quantum mechanics.

The different variants of deformations of the commutation relations arise in the description of the systems of particles with continuous interpolating (Bose and Einstein) statistics, the theory of fractional quantum Hall effect, high T_c superconductivity . The q-deformed oscillators are used widely in molecular and nuclear spectroscopy.

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In parallel with the q-deformed commutation relations the two-parameter (p, q)-deformatoin of these relations has been introduced in [4, 5]. The connection (p, q)-deformed oscillator algebra with (p, q)-hypergeometric functions has been established in [6].

Various generalizations of this deformation were considered by number of authors in the forthcoming period.

The two-parameter deformed boson algebra invariant under the quantum group SU_{q_1/q_2} ('*Fibonacci oscillator*') was studied in [7]. The multi-mode generalization of this algebra describes the two-parameter deformed boson gas. It shows Bose–Einstein condensation for low temperatures in the interval $q_2 > q_1 > 0$ [8], whereas for the value of $q_1^2 + q_2^2 \approx 4.16$ it behaves as a fermion gas for high temperatures.

In the phenomenological description of the particle physics the unusual behavior of the intercepts (or the strength) of the deformed bosons correlation functions within the q- and (pq)-Bose gas models have been studied [9, 10]. While the intercept of the two-particle correlation function is well described within the q-Bose gas model, the intercepts of n-order correlators require essentially the (pq)-Bose gas model description.

The multi-parameter deformed quantum algebras have been used in [11] to construct the integrable multi-parameter quantum spin chains. It is natural that the increase of the number of deformation parameters makes the method of the deformations more flexible. Although any multi-parameter deformed quantum algebra may be mapped onto a standard one parameter deformed algebra [12, 13] the physical results in both cases are not the same.

One of the methods of the construction of multi-parameter deformed oscillator algebras is the $(q; \alpha, \beta, \gamma)$-deformation of the one-parameter q-deformed oscillator algebras [14, 15].

The attractive properties of this deformation turn out to be useful in the area of nonlinear quantum optics and condensed matter physics. Nonlinear vector coherent states of the f-deformed spin–orbit Hamiltonians became the focus of attention of recent research. In [16] these states have been constructed for the $(p, q; \alpha, 0, l)$-deformed oscillator algebra [17].

The Hamiltonian of the electromagnetic monochromatic field in the Kerr medium [18] is embedded in the algebra of the four-parameter deformed oscillator [19]. This gives the complete description of the energy spectrum of this system. Furthermore, the modified oscillator algebra [20] has found applications in the study of the integrability of the two-particle Calogero model [21]. This algebra has been generalized to the C_λ-extended oscillator algebra [22]. Formalism of generalized modified S_N-extended oscillator algebras turned out to be relevant for solution of the many-anyon problem in 2+1 dimension [23].

With the hope to exploit such deformations for construction of new integrable models the 'hybrid' model [24] of the modified [20] and q-deformed [2, 3] oscillator algebras has been proposed.

To complete this cycle of ideas we consider the generalized $(q; \alpha, \beta, \gamma; \nu)$-deformed oscillator algebra as 'the synthesis' of the $(q; \alpha, \beta, \gamma)$-deformed [14, 15] and ν-modified oscillator algebras [20].

In section 2 some of the q-deformed oscillator algebras are placed in the order of their complication. In section 3 we find the structure function of $(q; \alpha, \beta, \gamma; \nu)$-deformed oscillator algebra. We show that this algebra is embedded into the deformed C_2-extended oscillator algebra [22]. In section 4 we give, following [22, 25], a classification of the representations of this algebra. In section 5 for the special choice of the deformation parameters of the algebra we study the oscillator [26] with the discrete spectrum of its 'quantized coordinate' Q. This operator is represented by a Jacobi matrix. We find the eigenvalues and eigenfunctions of this operator. The eigenfunctions are expressed by the (generalized) discrete q-Hermite I polynomial [27]. In section 6 we represent the Hamiltonian of this model as a function of the number operator and study its asymptotic energy spectrum behavior. Section 7 is devoted to discussion of these results and their possible applications.

2. The oscillator algebra and its generalized deformations

The oscillator algebra of the quantum harmonic oscillator is defined by canonical commutation relations

$$[a, a] = [a^+, a^+] = 0, \qquad [a, a^+] = 1, \qquad [N, a] = -a, \qquad [N, a^+] = a^+. \tag{1}$$

They allow for different types of deformations. Some of them have been called *generalized deformed oscillator algebras* [14, 28–30]. Each of them defines an algebra generated by elements (generators) $\{\mathbf{1}, a, a^+, N\}$ and relations

$$[N, a] = -a, \qquad [N, a^+] = a^+, \qquad a^+a = f(N), \qquad aa^+ = f(N+1), \tag{2}$$

where f is called *the structure function of the deformation*. Among them—the multiparameter generalization of one-parameter deformations [14, 15, 17, 19, 22, 24, 26, 30].

Let us recount some of them.

1. The Arik–Coon q-deformed oscillator algebra [1]
$$\begin{aligned} &aa^+ - qa^+a = 1, \qquad [N, a] = -a, \qquad [N, a^+] = a^+, \qquad q \in \mathbb{R}_+, \\ &f(n) = \frac{1 - q^n}{1 - q}. \end{aligned} \tag{3}$$
2. The Biedengarn–Macfarlane q-deformed oscillator algebra [2, 3]
$$\begin{aligned} &aa^+ - qa^+a = q^{-N}, \qquad aa^+ - q^{-1}a^+a = q^N, \qquad [N, a] = -a, \qquad [N, a^+] = a^+, \\ &f(n) = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad q \in \mathbb{R}_+. \end{aligned} \tag{4}$$
3. The Chung–Chung–Nam–Um generalized $(q; \alpha, \beta)$-deformed oscillator algebra [14]
$$\begin{aligned} &aa^+ - qa^+a = q^{\alpha N+\beta}, \qquad [N, a] = -a, \qquad [N, a^+] = a^+, \qquad q \in \mathbb{R}_+, \qquad \alpha, \beta \in \mathbb{R}, \\ &f(n) = \begin{cases} q^\beta \dfrac{q^{\alpha n} - q}{q^\alpha - q}, & \text{if} \quad \alpha \neq 1 \\ nq^{n-1+\beta}, & \text{if} \quad \alpha = 1. \end{cases} \end{aligned} \tag{5}$$
4. The generalized $(q; \alpha, \beta, \gamma)$-deformed oscillator algebra [15]
$$\begin{aligned} &aa^+ - q^\gamma a^+a = q^{\alpha N+\beta}, \qquad [N, a] = -a, \qquad [N, a^+] = a^+, \qquad q \in \mathbb{R}_+, \alpha, \beta, \gamma \in \mathbb{R}, \\ &f(n) = \begin{cases} q^\beta \dfrac{q^{\alpha n} - q^{\gamma n}}{q^\alpha - q^{\gamma n}}, & \text{if} \quad \alpha \neq \gamma \\ nq^{n-1+\beta}, & \text{if} \quad \alpha = \gamma. \end{cases} \end{aligned} \tag{6}$$
5. The ν-modified oscillator algebra [20, 21]
$$\begin{aligned} &[a, a^+] = 1 + 2\nu K, \qquad [N, a] = -a, \qquad [N, a^+] = a^+, \\ &aK = -Ka, \qquad a^+K = -Ka^+, \qquad K^2 = 1, \qquad \nu \in \mathbb{R}, \\ &f(n) = \begin{cases} 2k + 1 + 2\nu, & \text{if} \quad n = 2k \\ 2k + 2, & \text{if} \quad n = 2k + 1. \end{cases} \end{aligned} \tag{7}$$

This oscillator, as it has been shown in [21], is linked to the two-particle Calogero model [31].

6. The deformed C_λ-extended oscillator algebra [22] is defined by the relations

$$[a,a^+]_q \equiv aa^+ - qa^+a = H(N) + K(N)\sum_{k=0}^{\lambda-1}\nu_k P_k, \qquad [N,a]=-a, \qquad [N,a^+]=a^+,$$
$$aK = -Ka, \qquad a^+K = -Ka^+, \qquad K^2 = 1, \qquad \nu_k \in \mathbb{R}, \tag{8}$$

where $\nu_k \in \mathbb{R}$ and $H(K)$, $K(N)$ are real analytic functions. This algebra permits the two Casimir operators $C_1 = \mathrm{e}^{2\pi N}$ and $C_2 = \sum_{k=0}^{\lambda-1} \mathrm{e}^{-2\pi \mathrm{i}(N-k)/\lambda} P_k$.

7. The new $(q;\nu)$-deformed oscillator [24]

$$\begin{aligned} &aa^+ - qa^+a = (1+2\nu K)q^{-N}, \qquad [N,a]=-a,\\ &[N,a^+]=a^+, \qquad Ka=-aK, \qquad Ka^+=-a^+K, \qquad K^2=1,\\ &f(n) = \left(\frac{q^n-q^{-n}}{q-q^{-1}} + 2\nu\frac{q^n-(-1)^n q^{-n}}{q+q^{-1}}\right)\end{aligned} \tag{9}$$

has been defined by the combination of the idea of Biedenharn–Macfarlane [2, 3] q-deformation with the Brink, Hanson and Vasiliev idea [21] of the ν-modification of the oscillator algebra.

8. In order to complete this cycle of ideas we consider a $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra—'hybrid' of the $(q;\alpha,\beta,\gamma)$-deformed (6) and the ν-modified (7) oscillator algebras—or, more exactly, an oscillator defined by the generators $\{I, a, a^+, N, K\}$ and relations

$$\begin{aligned} &aa^+ - q^\gamma a^+a = (1+2\nu K)q^{\alpha N+\beta}, \qquad [N,a]=-a, \qquad [N,a^+]=a^+,\\ &Ka=-a, \qquad Ka^+=-a^+K, \qquad [N,K]=0, \qquad N^+=N, \qquad K^+=K,\end{aligned} \tag{10}$$

where $q \in \mathbb{R}_+, \alpha, \beta \in \mathbb{R}, \nu \in \mathbb{R}-\{0\}$. The oscillator algebra of this model unifies all deformations 1–7 of the oscillator algebra (1).

3. Generalized $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra and its simplest properties

(a) $(q;\alpha,\beta,\gamma;\nu)$-deformed structure function. Description of a deformed oscillator algebra requires the determination of the deformation structure function $f(n)$.

Equations (2) and (10) imply the recurrence relation

$$f(n+1) - q^\gamma f(n) = (1+2\nu(-1)^n)q^{\alpha n+\beta}. \tag{11}$$

Its solution is obtained by the method of mathematical induction

0. $f(1) = q^\gamma f(0) + (1+2\nu(-1)^0)q^{\alpha 0+\beta}$;

1. $f(2) = q^\gamma f(1) + (1-2\nu(-1)^1)q^{\alpha 1+\beta} = [q^{2\gamma} f(0) + (1+2\nu(-1)^0)q^\gamma q^{\alpha 0+\beta} + (1-2\nu(-1)^1)q^{\alpha 1+\beta}$;

2. $f(3) = q^\gamma f(2) + (1-2\nu(-1)^2)q^{\alpha 2+\beta} = q^{3\gamma} f(0) + (1+2\nu(-1)^0)q^{2\gamma} q^{\alpha 0+\beta} + (1-2\nu(-1)^1)q^\gamma q^{\alpha 1+\beta} + (1-2\nu(-1)^2)q^{\alpha 2+\beta}$;

3. $f(4) = q^\gamma f(3) + (1+2\nu(-1)^3)q^{\alpha 3+\beta} = q^{4\gamma} f(0) + (1+2\nu(-1)^0)q^{3\gamma} q^{\alpha 0+\beta} + (1-2\nu(-1)^1)q^{2\gamma} q^{\alpha 1+\beta} + (1-2\nu(-1)^2)q^\gamma q^{\alpha 2+\beta} + (1+2\nu(-1)^3)q^{\alpha 3+\beta}$;

$$\vdots$$

n. $f(n) = q^{n\gamma} f(0) + \sum_{k=0}^{n-1} q^{\gamma(n-k-1)} q^{\alpha k+\beta} + 2\nu \sum_{k=0}^{n-1} q^{\gamma(n-k-1)}(-1)^k q^{\alpha k+\beta}$..

The solution of equation (11) with the initial value $f(0)$ is given by the following formula:

$$f(n) = \begin{cases} f(0)q^{\gamma n} + q^{\beta}\left(\dfrac{q^{\gamma n} - q^{\alpha n}}{q^{\gamma} - q^{\alpha}} + 2\nu\dfrac{q^{\gamma n} - (-1)^n q^{\alpha n}}{q^{\gamma} + q^{\alpha}}\right), & \text{if} \quad \alpha \neq \gamma \\ f(0)q^{\gamma n} + nq^{\gamma(n-1)+\beta} + 2\nu q^{\gamma(n-1)+\beta}\left(\dfrac{1-(-1)^n}{2}\right), & \text{if} \quad \alpha = \gamma. \end{cases} \tag{12}$$

(b) Positivity of $(q; \alpha, \beta, \gamma; \nu)$-deformed structure function. We try to define the values of the parameters $\{\alpha, \beta, \gamma, \nu\}$, where function $f(n)$ is positive.

The inequality

$$\frac{q^{\gamma n} - q^{\alpha n}}{q^{\gamma} - q^{\alpha}} + 2\nu\frac{q^{\gamma n} - (-1)^n q^{\alpha n}}{q^{\gamma} + q^{\alpha}} > 0, \tag{13}$$

can be rewritten as

$$(q^{\gamma n} - q^{\alpha n})\left(\frac{1}{q^{\gamma} - q^{\alpha}} + 2\nu\frac{1}{q^{\gamma} + q^{\alpha}}\right) > 0 \tag{14}$$

for even n, and

$$\frac{q^{\gamma n} - q^{\alpha n}}{q^{\gamma n} + q^{\alpha n}}\frac{q^{\gamma} + q^{\alpha}}{q^{\gamma} - q^{\alpha}} + 2\nu > 0 \tag{15}$$

for odd n.

This implies that

$$1 + 2\nu > 0 \tag{16}$$

if $q > 1, \gamma - \alpha > 0 (q < 1, \gamma - \alpha < 0)$, and

$$-1 < 2\nu < -\frac{q^{\gamma} + q^{\alpha}}{q^{\gamma} - q^{\alpha}} \tag{17}$$

if $q < 1, \gamma - \alpha > 0 (q > 1, \gamma - \alpha < 0)$.

The conditions (16) and (17) are sufficient to ensure that the Fock representation of relation (10) has the Hermitian properties.

(c) Useful formulae. The following formulae will be useful for the study of this algebra. One of them is

$$a(a^+)^n - q^{\gamma n}(a^+)^n a = [n; \alpha, \gamma; \nu K](a^+)^{n-1}q^{\alpha N+\beta}, \tag{18}$$

where $n \geqslant 1$, and the other one

$$[n; \alpha, \gamma; \nu K] = \begin{cases} \left(\dfrac{q^{\gamma n} - q^{\alpha n}}{q^{\gamma} - q^{\alpha}} + 2\nu K\dfrac{q^{\gamma n} - (-1)^n q^{\alpha n}}{q^{\gamma} + q^{\alpha}}\right), & \text{if} \quad \alpha \neq \gamma; \\ nq^{\alpha(n-1)} + 2\nu K q^{\alpha(n-1)}\left(\dfrac{1-(-1)^n}{2}\right), & \text{if} \quad \alpha = \gamma \end{cases} \tag{19}$$

is deduced by the method of mathematical induction.

Indeed, for $n = 1$ relation (18) is true by definition. The assumption that it is true for some n implies

$$\begin{aligned} \{a(a^+)^{n+1} &= a(a^+)^n a^+ = (q^{\gamma n}(a^+)^n a + [n; \alpha, \gamma; \nu K](a^+)^{n-1}q^{\alpha N+\beta})a^+ \\ &= q^{\gamma n}(a^+)^n aa^+ + (a^+)^n[n; \alpha, \gamma; -\nu K]q^{\alpha(N+1)+\beta} \\ &= q^{\gamma n}(a^+)^n(q^{\gamma}a^+a + (1+2\nu K)q^{\alpha N+\beta}) + (a^+)^n[n; \alpha, \gamma; -\nu K]q^{\alpha(N+1)+\beta} \\ &= q^{\gamma(n+1)}(a^+)^{(n+1)}a + q^{\gamma n}(a^+)^n(1+2\nu K)q^{\alpha N+\beta} + q^{\alpha}(a^+)^n[n; \alpha, \gamma; -\nu K]q^{\alpha N+\beta}. \end{aligned}$$

The direct calculations leads to (18).

The second formula gives the generated function for $[n;\alpha,\gamma;\nu K]$:

$$\sum_{n=0}^{\infty}[n;\alpha,\gamma;\nu K]z^n = \begin{cases} \dfrac{z}{1-q^{\gamma}z}\left(\dfrac{1}{1-q^{\alpha}z}+2\nu K\dfrac{1}{1+q^{\alpha}z}\right), & \text{if} \quad \alpha\neq\gamma \\ \dfrac{z}{(1-q^{\gamma}z)^2}+2\nu K\dfrac{z}{1-q^{2\gamma}z^2}, & \text{if} \quad \alpha=\gamma. \end{cases} \tag{20}$$

(d) Deformed C_2-extended and $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebras. The defining relations of the deformed C_2-extended oscillator are given by

$$\begin{aligned} &[N,a^+]=a^+, \qquad [N,P_k]=0, \qquad a^+P_k=P_{k+1}a^+, \qquad P_1+P_2=I, \\ &P_kP_l=\delta_{k,l}P_l, \qquad aa^+-q^{\gamma}a^+a=H(N)+\nu(E(N+1)+q^{\gamma}E(N))(P_0-P_1), \end{aligned} \tag{21}$$

where $q,\nu\in\mathbb{R}, k,l=1,2$, and $E(N), H(N)$ are real analytic functions. As we saw above the deformed extended oscillator algebra C_λ permits the two Casimir operators C_1, C_2. In case of the C_2-extended oscillator algebra they have the form

$$C_1=\mathrm{e}^{2\pi\mathrm{i}N}, \qquad C_2=\Sigma_{k=0}^{1}\mathrm{e}^{-2\pi\mathrm{i}(N-k)/2}=\mathrm{e}^{\mathrm{i}\pi N}K. \tag{22}$$

Let us define the operator

$$\tilde{C}_3=q^{-\gamma N}(D(N)+\nu E(N)K-a^+a), \tag{23}$$

where $D(N), E(N)$ are some analytic functions of N. The operator $\tilde{C}_3$ will be the Casimir operator of the oscillator algebra (21) if the only one condition $[\tilde{C}_3,a]=0$ holds. It amounts to determination of the solution of the equations

$$D(N+1)-q^{\gamma}D(N)=H(N), \qquad E(N+1)\beta_{k+1}-q^{\gamma}E(N)\beta_k=K(N)\nu_k, \tag{24}$$

where $\nu_0=-\nu_1=\nu, \beta_0=0, \beta_2=0, \beta_1=\nu, \quad k=0,1$. Substituting the solution $E(N)=2q^{\alpha N+\beta}/(q^{\gamma}+q^{\alpha})$ of equation of (24) and $H(N)=q^{\alpha N+\beta}$ in (21), we obtain the commutation relations of the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra (10). Moreover, the solution

$$D(N)=\begin{cases} q^{\beta}\left(\dfrac{q^{\gamma N}-q^{\alpha N}}{q^{\gamma}-q^{\alpha}}+2\nu\dfrac{q^{\gamma N}}{q^{\gamma}+q^{\alpha}}\right) & \text{if} \quad \gamma\neq\alpha \\ q^{\beta}(q^{\gamma(N-1)}N+\nu q^{-\gamma}) & \text{if} \quad \gamma=\alpha \end{cases}$$

of the first equation (24) gives the explicit form of the Casimir operator

$$\tilde{C}_3=\begin{cases} q^{-\gamma N}\left(\left(\dfrac{q^{\gamma N}-q^{\alpha N}}{q^{\gamma}-q^{\alpha}}+2\nu\dfrac{q^{\gamma N}-(-1)^Nq^{\alpha N}}{q^{\gamma}+q^{\alpha}}\right)q^{\beta}-a^+a\right) & \text{if} \quad \alpha\neq\gamma \\ q^{-\gamma N}(N+\nu(1+(-1)^N)q^{\gamma N+\beta}-a^+a) & \text{if} \quad \alpha=\gamma. \end{cases} \tag{25}$$

4. Classification of representations of the unified $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra

As it has been shown in the previous section the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator algebra allows for a nontrivial center which means that it has irreducible non-equivalent representations [32, 33]. We give a classification of these representations by a method similar to that in the articles [25, 34].

Due to relations (10) there exists a vector $|0\rangle$ such that

$$a^+a|0\rangle=\lambda_0|0\rangle, \qquad aa^+|0\rangle=\mu_0|0\rangle, \qquad N|0\rangle=\varkappa_0|0\rangle, \qquad K|0\rangle=\omega\,\mathrm{e}^{-\mathrm{i}\pi\varkappa_0}|0\rangle,$$

where ω is the value of the Casimir operator C_2 on the given irreducible representation. By using formula (18) we find that vectors

$$|n\rangle = \begin{cases} (a^+)^n|0\rangle, & \text{if} \quad n \geqslant 0 \\ (a)^{-n}|0\rangle, & \text{if} \quad n < 0 \end{cases} \tag{26}$$

are eigenvectors of the operators a^+a and aa^+

$$a^+a|n\rangle = \lambda_n|n\rangle, \qquad aa^+|n\rangle = \mu_n|n\rangle.$$

Let us define a new system of the orthonormal vectors $\{|n\rangle\}_{n=-\infty}^{n=\infty}$, by

$$|n\rangle = \begin{cases} \left(\prod_{k=1}^{n} \lambda_k\right)^{-1/2} (a^+)^n|0\rangle, & \text{if} \quad n \geqslant 0 \\ \left(\prod_{k=1}^{-n} \lambda_{n+k}\right)^{-1/2} (a)^{-n}|0\rangle, & \text{if} \quad n < 0. \end{cases} \tag{27}$$

Then relations (10) are represented by the operators

$$\begin{aligned} a^+|n\rangle &= \sqrt{\lambda_{n+1}}|n+1\rangle, \qquad & a|n\rangle &= \sqrt{\lambda_n}|n-1\rangle, \\ N|n\rangle &= (\varkappa_0 + n)|n\rangle, \qquad & K|n\rangle &= \frac{(-1)^n}{2\nu}B|n\rangle, \end{aligned} \tag{28}$$

where $B = 2\nu\omega\,\mathrm{e}^{-\mathrm{i}\pi\varkappa_0} \in \mathbb{R}$. Due to non-negativity of the operators a^+a, aa^+ we have $\lambda_n \geqslant 0$ and $\mu_n \geqslant 0$. By using (10), we find (28) and when $\lambda_n = \mu_{n-1}$ we obtain the recurrence relation

$$\lambda_{n+1} - q^\gamma \lambda_n = (1 + (-1)^n B) q^{\alpha(n+\varkappa_0)+\beta}. \tag{29}$$

Taking into account relation (12) the solution of equation (29) can be represented by

$$\lambda_n = \begin{cases} \lambda_0 q^{\gamma n} + q^{\alpha\varkappa_0+\beta}\left(\dfrac{q^{\gamma n} - q^{\alpha n}}{q^\gamma - q^\alpha} + B\dfrac{q^{\gamma n} - (-1)^n q^{\alpha n}}{q^\gamma + q^\alpha}\right), & \text{if} \quad \alpha \neq \gamma; \\ \lambda_0 q^{\gamma n} + n q^{\gamma(n+\varkappa_0-1)+\beta} + B q^{\gamma(n+\varkappa_0-1)+\beta}\left(\dfrac{1-(-1)^n}{2}\right), & \text{if} \quad \alpha = \gamma. \end{cases} \tag{30}$$

Equivalently, using the expression of the Casimir operator (25) and the representation (28) the solution λ_n of equation (29) can be presented by means of the eigenvalue c_3 of the Casimir operator C_3 in irreducible representation of relation (10)

$$\lambda_n = \begin{cases} -q^{\gamma n} c_3 + q^\beta\left(\dfrac{q^{\gamma(n+\varkappa_0)} - q^{\alpha(n+\varkappa_0)}}{q^\gamma - q^\alpha} + 2\nu\dfrac{q^{\gamma(n+\varkappa_0)} - (-1)^{n+\varkappa_0} q^{\alpha(n+\varkappa_0)}}{q^\gamma + q^\alpha}\right), & \text{if} \quad \alpha \neq \gamma; \\ \lambda_0 q^{\gamma n} c_3 + n q^{\gamma(n+\varkappa_0-1)+\beta} + B q^{\gamma(n+\varkappa_0-1)+\beta}\left(\dfrac{1-(-1)^n}{2}\right), & \text{if} \quad \alpha = \gamma. \end{cases} \tag{31}$$

It is easy see that λ_n in (30) coincides with the value of (31), where $B = (-1)^{\varkappa_0} 2\nu$.

According to [22] representations of the generalized oscillator algebra are reduced to the four classes of unireps:

1. Representations *bounded from below*. They are defined by (28) and $n_1 \in \mathbb{Z}_{\leqslant 0}$ such that

$$\lambda_n = \begin{cases} \lambda_{n_1} = 0 & \text{if} \quad n_1 \in \{\ldots, -2, -1, 0\} \\ \lambda_n > 0 & \text{if} \quad n \in \{n_1 + 1, n_1 + 2, \ldots\}. \end{cases} \tag{32}$$

(i) Let λ_n be defined by formula (30) for $\gamma - \alpha = 0$, $q > 0$. The non-negativity of λ_n implies

$$\lambda_n = \lambda_0 + nq^{\gamma(\varkappa_0-1)+\beta} + Bq^{\gamma(\varkappa_0-1)+\beta}\left(\frac{1-(-1)^n}{2}\right) \geqslant 0. \tag{33}$$

From this one finds (32), and due to (28) $a|n_1\rangle = 0$.

After possible renumbering of vectors $|n\rangle$ we obtain $a|0\rangle = 0$ and due to (28) one gets $\lambda_0 = 0$. Therefore the representations are given by formulae (28) with

$$\lambda_n = nq^{\gamma(n+\varkappa_0-1)+\beta} + Bq^{\gamma(n+\varkappa_0-1)+\beta}\left(\frac{1-(-1)^n}{2}\right), \tag{34}$$

where $n \geqslant 0$. The arbitrary values of the parameter $\varkappa_0$, and $B \geqslant 0$ correspond to non-equivalent representations of (10).

(ii) Let λ_n be defined by (30) for $\gamma - \alpha > 0, q > 1 (\gamma - \alpha < 0, 0 < q < 1)$.

The assumption of the positivity of at least one of the numbers $\frac{1}{q^\gamma - q^\alpha} \pm \frac{B}{q^\gamma + q^\alpha}$ and the non-negativity of λ_n

$$\left(\lambda_0 q^{-(\alpha\varkappa+\beta)} + \frac{1}{q^\gamma - q^\alpha} + \frac{B}{q^\gamma + q^\alpha}\right) \geqslant q^{-(\gamma-\alpha)n}\left(\frac{1}{q^\gamma - q^\alpha} + \frac{(-1)^n B}{q^\gamma + q^\alpha}\right) \tag{35}$$

for $n \geqslant 0$ implies (32), and due to (28) $a|n_1\rangle = 0$.

After possible renumbering of the vectors $|n\rangle$ we obtain $a|0\rangle = 0$ and due to (28) one gets $\lambda_0 = 0$. Therefore the representations are given by (28) with

$$\lambda_n = q^{\alpha\varkappa_0+\beta+n\gamma}\left(\frac{1-q^{(\alpha-\gamma)n}}{q^\gamma - q^\alpha} + B\frac{1-(-1)^n q^{(\alpha-\gamma)n}}{q^\gamma + q^\alpha}\right), \qquad n \geqslant 0. \tag{36}$$

The non-negativity condition for λ_n implies $B \geqslant -1$. The arbitrary values of the parameter $\varkappa_0$, and $\lambda_0 = 0$ and $B > -1$ defined non-equivalent infinite-dimensional representations of (10).

(iii) Let λ_n be given by (30) for $\gamma - \alpha > 0, 0 < q < 1 (\gamma - \alpha < 0, q > 1)$.

It can be represented by

$$\begin{aligned}\lambda_n = q^{\alpha\varkappa_0+\beta+\gamma n}\Bigg\{&\left(\lambda_0 q^{-(\alpha\varkappa+\beta)} + \frac{1}{q^\gamma - q^\alpha} + \frac{B}{q^\gamma + q^\alpha}\right)\\ &- q^{-(\gamma-\alpha)n}\left(\frac{1}{q^\gamma - q^\alpha} + \frac{(-1)^n B}{q^\gamma + q^\alpha}\right)\Bigg\}.\end{aligned} \tag{37}$$

The non-negativity of $\lambda_n \geqslant 0$, negativity of $\left(\lambda_0 q^{-(\alpha\varkappa+\beta)} + \frac{1}{q^\gamma - q^\alpha} + \frac{B}{q^\gamma + q^\alpha}\right) < 0$, and the non-positivity of $\left(\frac{1}{q^\gamma - q^\alpha} + \frac{(-1)^n B}{q^\gamma + q^\alpha}\right)$ implies (32) and $a|n_1\rangle = 0$. The same arguments as in the items (i), (ii) give $a|0\rangle = 0$ and due to (28) $\lambda_0 = 0$. Therefore the representation is given by (28) with λ_n as in (36). The non-negativity condition for λ_n gives a restriction for possible values of B

$$-1 \leqslant B < -\frac{q^\gamma + q^\alpha}{q^\gamma + q^\alpha}. \tag{38}$$

The arbitrary values of the parameter $\varkappa_0$, and $-1 \leqslant B < -\frac{q^\gamma + q^\alpha}{q^\gamma + q^\alpha}$ and $\lambda_0 = 0$ distinguish irreducible representation of relations (10).

2. Representations *bounded from above.* They are defined by formulae (28) and $n_2 \in \mathbb{N}$ such that

$$\lambda_n = \begin{cases}\lambda_{n_2} = 0 & \text{if} \quad n_2 \in \{1, 2, 3, \ldots\}\\ \lambda_n > 0 & \text{if} \quad n \in \{n_2 - 1, n_2 - 2, \ldots\}.\end{cases} \tag{39}$$

Let λ_n be given by formula (30) $0 < q < 1, (\alpha - \gamma) > 0 \qquad (q > 1, (\alpha - \gamma) < 0)$ and both values $\frac{1}{q^\gamma - q^\alpha} \pm \frac{B}{q^\gamma + q^\alpha}$ are non-positive (at least one of them must be strictly negative).

The non-negativity condition $\lambda_n \geqslant 0$ implies the existence of n_2 such that (39) holds. From formulae (28) we obtain $a^+|n_2\rangle = 0$, and after possible renumbering we may assume that $a^+|0\rangle = 0$, and due to (28) $\lambda_1 = 0$.

The formula (33) implies $\lambda_0 = -q^{\alpha\varkappa_0+\beta-\gamma}(1+B)$. The condition $\lambda_0 \geqslant 0$ is equivalent to $B \leqslant -1$. It leads to

$$\lambda_n = q^{\alpha\varkappa_0+\beta+\gamma n}\left(\left(\frac{1-q^{(\alpha-\gamma)n}}{q^\gamma-q^\alpha} - q^{-\gamma}\right) + B\left(\frac{1-q^{(\alpha-\gamma)n}(-1)^n}{q^\gamma+q^\alpha} - q^{-\gamma}\right) \geqslant 0, \tag{40}$$

$n \leqslant 0$. Therefore the representation is given by formulae (28) with λ_n as in (40). If $B < -1$ the non-negativity condition for λ_n gives a restriction for possible values of B

$$B \leqslant -\frac{q^\gamma+q^\alpha}{q^\gamma+q^\alpha}. \tag{41}$$

The arbitrary values of parameter $\varkappa_0$ and $\lambda_0 = -q^{\alpha\varkappa_0+\beta-\gamma}(1+B)$, $B < \frac{q^\gamma+q^\alpha}{q^\gamma+q^\alpha}$ distinguish irreducible representation of (10).

3. The *finite-dimensional representation.* They are defined by formulae (28) and $n_1 \in \{\ldots,-2,-1,0\},$, $n_2 \in \{1,2,3,\ldots\}$ such that

$$\lambda_n = \begin{cases} \lambda_{n_1} = \lambda_{n_2} = 0 & \text{if} \quad n_2 \in \{1,2,3,\ldots\}, n_1 \in \{\ldots,-2,-1,0\} \\ \lambda_n > 0 & \text{if} \quad n \in \{n_1+1, n_1+2, \ldots, n_2-1\}. \end{cases} \tag{42}$$

(i) One-dimensional representations. These representations are given by formulae (28) with λ_n calculated from (36), (37) and (40) at $B=-1$

$$\lambda_n = q^{-\alpha\varkappa_0+\beta}\left(\frac{q^{\gamma n}-q^{\alpha n}}{q^\gamma-q^\alpha} - \frac{q^{\gamma n}-(-1)^n q^{\alpha n}}{q^\gamma+q^\alpha}\right). \tag{43}$$

In this case $n_1 = 0 (n_1 = -1)$ and $n_2 = 1 (n_2 = 0)$ and the representations (28) are defined by

$$a = a^+ = 0, \qquad N = \varkappa_0, \qquad K = -\frac{1}{2\nu}. \tag{44}$$

These representations are one dimensional. They are parametrized by $\varkappa_0$.

(ii) Two-dimensional representations. Let λ_n be defined (30) for $\gamma - \alpha > 0, 0 < q < 1, (\gamma - \alpha < 0, q > 1)$. The values of $\lambda_n, n \geqslant 0$ in (37) are non-negative if we assume $\left(\lambda_0 q^{-(\alpha\varkappa+\beta)} + \frac{1}{q^\gamma-q^\alpha} + \frac{B}{q^\gamma+q^\alpha}\right) = 0$ and both values $\left(\frac{1}{q^\gamma-q^\alpha} + \frac{(-1)^n B}{q^\gamma+q^\alpha}\right)$ are non-positive. At $B = \pm\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$ we obtain

$$\lambda_n = \frac{q^{\alpha(\varkappa_0+n)+\beta}}{q^\alpha-q^\gamma}(1\pm(-1)^n). \tag{45}$$

The value $\lambda_n = \frac{q^{\alpha(\varkappa_0+n)+\beta}}{q^\alpha-q^\gamma}(1+(-1)^n))$ implies (42) for $n_1 = -1, n_2 = 1$ and $\lambda_0 = \frac{2q^{\alpha\varkappa_0+\beta}}{q^\alpha-q^\gamma}$. The vector space of this representation is a span of the two-dimensional vectors

$$\begin{pmatrix}\psi_{-1}\\ \psi_0\end{pmatrix}$$

and due to (28) the representations are defined by

$$a = \begin{pmatrix} 0 & \sqrt{\frac{2q^{\alpha\varkappa_0+\beta}}{q^\alpha-q^\gamma}} \\ 0 & 0 \end{pmatrix}, \qquad a^+ = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{2q^{\alpha\varkappa_0+\beta}}{q^\alpha-q^\gamma}} & 0 \end{pmatrix},$$
$$N = \begin{pmatrix} \chi_0-1 & 0 \\ 0 & \chi_0 \end{pmatrix}, \qquad K = \frac{1}{2\nu}\frac{q^\gamma+q^\alpha}{q^\alpha-q^\gamma}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{46}$$

These representations are distinguished by the arbitrary values $\varkappa_0$, and $B = \pm\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$, $\lambda_0 = \frac{2q^{\alpha(\varkappa_0)+\beta}}{q^\alpha-q^\gamma}$.

The value $\lambda_n = \frac{q^{\alpha(\varkappa_0+n)+\beta}}{q^\alpha-q^\gamma}(1-(-1)^n)$ implies (42) for $n_1 = 0, n_2 = 2$. Moreover $\lambda_1 = \frac{2q^{\alpha(\varkappa_0+1)+\beta}}{q^\alpha-q^\gamma}$.

The vector space of this representation spanned by the two-dimensional vectors

$$\begin{pmatrix}\psi_0\\ \psi_{-1}\end{pmatrix}$$

and due to (28) the representations are given by

$$\begin{aligned}
a &= \begin{pmatrix}0 & 0\\ \sqrt{\frac{q^{\alpha(\varkappa_0+1)+\beta}}{q^\alpha-q^\gamma}} & 0\end{pmatrix}, \qquad a^+ = \begin{pmatrix}0 & \sqrt{\frac{q^{\alpha(\varkappa_0+1)+\beta}}{q^\alpha-q^\gamma}}\\ 0 & 0\end{pmatrix},\\
N &= \begin{pmatrix}\chi_0 & 0\\ 0 & \chi_0+1\end{pmatrix}, \qquad K = \frac{1}{2\nu}\frac{q^\gamma+q^\alpha}{q^\alpha-q^\gamma}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.
\end{aligned} \tag{47}$$

These representations are defined by arbitrary values of $\varkappa_0$, and $\lambda_0 = 0$, $B = -\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}$.

4. *Unbounded representations.* They are defined by (28) with $\lambda_n \geqslant 0$ for all $n \in \mathbb{Z}$. As it follows from (37) they are realized if $0 < q < 1$, $(\alpha-\gamma) > 0$ ($q > 1$, $(\alpha-\gamma) < 0$) and

$$\lambda_0 q^{-\alpha\varkappa_0-\beta} + \frac{1}{q^\gamma-q^\alpha} + \frac{B}{q^\alpha+q^\gamma} \geqslant 0, \qquad |B| < -\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}, \tag{48}$$

or

$$\lambda_0 q^{-\alpha\varkappa_0-\beta} + \frac{1}{q^\gamma-q^\alpha} + \frac{B}{q^\alpha+q^\gamma} > 0, \qquad |B| = -\frac{q^\gamma+q^\alpha}{q^\gamma-q^\alpha}. \tag{49}$$

Under the transformations

$$\begin{aligned}
&B \to B' = (-1)^n B, \qquad \varkappa_0 \to \varkappa' = \varkappa + n,\\
&\lambda_0 \to \lambda_0' = \lambda_0 q^{\gamma n} + q^{\alpha\varkappa_0+\beta}\left(\frac{q^{\gamma n}-q^{\alpha n}}{q^\gamma-q^\alpha} + B\frac{q^{\gamma n}-(-1)^n q^{\alpha n}}{q^\gamma+q^\alpha}\right)
\end{aligned} \tag{50}$$

the conditions (48) and (49) are preserved

$$\lambda_n' = \lambda_0 q^{2\gamma n} + q^{\alpha\varkappa_0+\beta}\left(\frac{q^{2\gamma n}-q^{2\alpha n}}{q^\gamma-q^\alpha} + B\frac{q^{2\gamma n}-q^{2\alpha n}}{q^\gamma+q^\alpha}\right), \tag{51}$$

$$\lambda_0' q^{-\alpha\varkappa_0'-\beta} + \frac{1}{q^\gamma-q^\alpha} + \frac{B'}{q^\alpha+q^\gamma} = q^{(\gamma-\alpha)n}\left(\lambda_0 q^{-\alpha\varkappa_0-\beta} + \frac{1}{q^\gamma-q^\alpha} + \frac{B}{q^\alpha+q^\gamma}\right) \tag{52}$$

and *unbounded representations* are transformed into themselves.

5. Spectrum of 'quantized coordinate' *Q*

According to the previous section if we put $\omega = 1$, $\varkappa_0 = 0$, in (36) we obtain the Fock representation of relations (10)

$$\begin{aligned}
&N|n\rangle = n|n\rangle, \qquad K|n\rangle = (-1)^n|n\rangle,\\
&a^+|2n\rangle = q^{\beta/2}\left(\frac{q^{\gamma(2n+1)}-q^{\alpha(2n+1)}}{q^\gamma-q^\alpha} + 2\nu\frac{q^{\gamma(2n+1)}+q^{\alpha(2n+1)}}{q^\gamma+q^\alpha}\right)^{1/2}|2n+1\rangle,\\
&a^+|2n-1\rangle = q^{\beta/2}\left(\frac{q^{2\gamma n}-q^{2\alpha n}}{q^\gamma-q^\alpha} + 2\nu\frac{q^{2\gamma n}-q^{2\alpha n}}{q^\gamma+q^\alpha}\right)^{1/2}|2n\rangle.
\end{aligned} \tag{53}$$

The properties of the quantum harmonic oscillator are closely related to Hermite polynomials. Different q-deformed oscillators generate distinct q-deformed Hermite polynomials and their generalization. Unlike the quantum harmonic oscillator, generalized q-deformed oscillators frequently define the position and momentum operators with the discrete spectrum, and their eigenfunctions are represented by means of a q-Hermite polynomials. For some q-deformed oscillators this problem was studied in [35–37]. In [26] the generalized q-deformed oscillators connected with the discrete (generalize) q-Hermite I and q-Hermite II polynomials have been build. In this section we show that corresponding oscillator algebras of these models are embedded in the $(q; \alpha, \beta, \gamma : \nu)$-deformed oscillator algebra.

Let us consider the operator ('quantized coordinate') $Q = a^{+} + a$, or

$$Q|n\rangle = r_n|n+1\rangle + r_{n-1}|n-1\rangle, \qquad r_n = f^{1/2}(n+1). \tag{54}$$

The self-adjointness and spectral properties of this operator is defined by polynomials $P_n(x)$ of first kind for the Jacobi matrix Q.

Defining the generalized eigenfunction $Q|x\rangle = x|x\rangle$, where $|x\rangle = \sum_{n=0}^{\infty} P_n(x)|n\rangle$, we obtain the recurrence relation

$$f^{1/2}(n)P_{n-1}(x) + f^{1/2}(n+1)P_{n+1}(x) = xP_n(x), \tag{55}$$

where $P_{-1}(x) = 0$, $P_0 = 1$.

In this section we consider the oscillator (10) for special choice of parameters

$$\alpha = 2a, \qquad \gamma = 2a + c - 1 \qquad \beta = 2a + b, \nu = 0. \tag{56}$$

To the end of this section the symbol 'a' is used to denote both the parameter deformation and the annihilation operator. Its meaning will be clear from the context.

We obtain relation [26]

$$aa^{+} - q^{2a+c-1}a^{+}a = q^{2a(N+1)+b}, \tag{57}$$

or equivalent

$$aa^{+} - q^{2a}a^{+}a = q^{2a(N+1)+b}q^{(c-1)N}, \tag{58}$$

which together with other relations of (10) form the corresponding oscillator algebra.

The self-adjoints of the operator (54) can be established by theorem 1.3 chapter VII in [38]. For this purpose it is sufficient to prove the divergence of the series

$$\sum_{n=0}^{\infty} \frac{1}{r_n} = \infty. \tag{59}$$

The operator Q (54) is a symmetric one. If

$$r_n \geqslant 0, \qquad r_{n-1}r_{n+1} \leqslant r_n^2, \qquad \sum_{n=0}^{\infty} \frac{1}{r_n} < \infty. \tag{60}$$

by theorem 1.5 chapter VII in [38] its closure $\bar{Q}$ is not a self-adjoint operator.

In our case the property $r_n \geqslant 0, r_{n-1}r_{n+1} \leqslant r_n^2$ is satisfied automatically, therefore, the self-adjointness of the operator Q are reduced to the proving of the convergence of the series (60). We consider the case

$$q < 1, \qquad \begin{cases} a < 0, & c-1 > 0, \quad \text{divergent}, \\ a > 0, & c-1 > 0, \quad \text{convergent}, \\ a + \dfrac{c-1}{2} < 0, & c-1 < 0, \quad \text{divergent}, \\ a + \dfrac{c-1}{2} > 0, & c-1 < 0, \quad \text{convergent}. \end{cases} \tag{61}$$

Equation (55) for this case takes the form

$$x P_n(x;q) = \left(\frac{1}{1-q'}\right)^{1/2} q^{an+b/2}(1-q'^n)^{1/2} P_{n-1}(x;q) + \left(\frac{1}{1-q'}\right)^{1/2} q^{a(n+1)+b/2}(1-q'^{(n+1)})^{1/2} P_{n+1}(x;q), \tag{62}$$

where $q' = q^{c-1}$. Change and rescale of the variables $y = (1-q')^{1/2}x$, $P_n(x;q) = \psi_n((1-q')^{1/2}x;q)$ then it results in

$$x\psi_n(x;q) = q^{a(n+1)+b/2}(1-q'^n)^{1/2}\psi_{n+1}(x;q) + q^{an+b}(1-q'^n)^{1/2}\psi_{n-1}(x;q). \tag{63}$$

Representing $\psi_n(x;q)$ as

$$\psi_n(x;q) = \frac{q^{an^2/2}}{q^{(a+b)n/2}(q';q')_n^{1/2}} h_n(x;q)$$

we get from (63) the recurrence relation for the (generalized) discrete q-Hermite I polynomials $h_n(x;q)$

$$xh_n(x;q) = h_{n+1}(x;q) + q^{2an+b}(1-q'^n)h_{n-1}(x;q). \tag{64}$$

The solution of this equation is given by anzatz

$$h_n(x;q) = \sum_{k=0}^{[n/2]} \frac{(q';q')_n}{((a_n,c_n);(1,q'^d))_k}(-1)^k q^{(2an+b)k} q'^{k(k-n)} x^{n-2k} \tag{65}$$

where a_n, c_n, d are unknown quantities. The notation

$$((a,c);(p,q))_k = \begin{cases} 1, & \text{if} \quad k=0; \\ (a-c)(ap-cq)\cdots(ap^{k-1}-cq^{k-1}), & \text{otherwise} \end{cases} \tag{66}$$

is used according to [39]. The substitution of (65) into (64) leads to the equation

$$1 - q'^{n+1} - q^{2an+b}(a_n - c_n q'^{d(k-1)})q'^{-2(k-1)} = 1 - q'^{n-2k+1}. \tag{67}$$

The solutions of this equation

$$a_n = q^{-2an-b}q'^{n-1}, \qquad c_n = q^{-2an-b}q'^{n+1}, \quad d+2 \tag{68}$$

give the solutions of (64)

$$h_n(x;q) = \sum_{k=0}^{[n/2]} \frac{(q';q')_n}{(q'^2;q'^2)_k(q';q')_{n-2k}}(-1)^k q^{(2an+b)k} q'^{k(k-n)} x^{n-2k}. \tag{69}$$

They can be represented in terms of the basic hypergeometric function as

$$h_n(x;q) = x^n {}_2\phi_0\left(\begin{matrix} q'^{-n}, & q'^{-n+1} \\ & - \end{matrix} \,\middle|\, q'^2; \frac{q^{2an+b}q'^n}{x^2}\right). \tag{70}$$

As it has been shown in [26] this solution at $a = \frac{1}{2}, b = -1, c = 2$ is reduced to discrete q-Hermite I polynomials, and relations (57) (or (58)) are reduced to the equation of the corresponding q-deformed oscillator.

The solutions $P_n(x;q)$ of equations (62) with the initial conditions $P_{-1}(x;q) = 0$, $P_0(x;q) = 1$ can be written as polynomials of degree n in x

$$P_n(x;q) = \frac{q^{-an^2/2}}{q^{\frac{a+b}{2}n}(q';q')_n^{1/2}} h_n(\sqrt{1-q'}x;q). \tag{71}$$

Now we restrict ourselves by the condition $a = (c-1)/2$ in (71). Then

$$P_n(x;q) = \frac{q^{-an(n-1)/2}}{(q';q')_n^{1/2}} h_n^0(q^{-(2a+b)/2}\sqrt{1-q'}x;q'). \tag{72}$$

These polynomials are orthogonal with respect to the discrete measure

$$\begin{aligned} d\omega(x) = {} & \frac{q^{-(2a+b)}\sqrt{1-q'}}{2}(q';q'^2)_\infty \delta\left(x - \frac{q'^0}{q^{-(2a+b)}\sqrt{1-q'}}\right) dx \\ & + \sum_{k>0} \frac{q^{-(2a+b)/2}\sqrt{1-q'}|x|}{2} \frac{(q^{-(2a+b)}(q'^2(1-q')x^2, q'; q^2)_\infty}{(q';q')_\infty} \\ & \times \delta\left(x - \frac{q'^k}{q^{-(2a+b)/2}\sqrt{1-q'}}\right) dx \\ & + \sum_{k>0} \frac{q^{-(2a+b)/2}\sqrt{1-q'}|x|}{2} \frac{(q^{-(2a+b)}(q'^2(1-q')x^2, q'; q^2)_\infty}{(q';q')_\infty} \\ & \times \delta\left(x + \frac{q'^k}{q^{-(2a+b)/2}\sqrt{1-q'}}\right) dx. \end{aligned} \tag{73}$$

The orthogonality relation has the form

$$\begin{aligned} \frac{\delta_{mn}}{(q';q')_n} = {} & \frac{1}{2}\frac{(q';q')_\infty}{(q'^2;q'^2)_\infty} P_m(1;q)P_n(1;q) \\ & + \sum_{k>0}\{P_m(q'^k;q)P_n(q'^k;q) + P_m(-q'^k;q)P_n(-q'^k;q)\} \\ & \times \frac{q'^k}{2}\frac{(q'^{2k+2}, q'; q'^2)_\infty (q';q'^2)_\infty}{(q'^2;q'^2)_\infty}. \end{aligned} \tag{74}$$

It follows that spectrum of the position operator Q is

$$SpQ = \left\{ \frac{\pm q^{(2a+b)/2}}{\sqrt{1-q'}}, \frac{\pm q^{(2a+b)/2}q'}{\sqrt{1-q'}}, \ldots, \frac{\pm q^{(2a+b)/2}q'^k}{\sqrt{1-q'}}, \ldots; k \geqslant 0 \right\}. \tag{75}$$

6. Hamiltonian and energy spectrum of the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator

(g) Energy spectrum of the free $(q;\alpha,\beta,\gamma;\nu)$*-deformed oscillator.* In general, a Hamiltonian $H(a^+, a, N)$ of an oscillator is a function of the operators a^+, a, N. As an example, we consider the free standard Hamiltonian

$$H = \frac{\hbar\omega_0}{2}\{a^+, a\} = \frac{\hbar\omega_0}{2}(aa^+ + a^+a) \tag{76}$$

of the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator.

Due to the structure function of deformation the free Hamiltonian H of the $(q;\alpha,\beta,\gamma;\nu)$-deformed oscillator in the Fock representation (53) can be written only as a function of the number operator N

$$\begin{aligned} H = \frac{\hbar\omega_0}{2}q^\beta \Bigg\{ & \left(\frac{q^{\gamma N} - q^{\alpha N}}{q^\gamma - q^\alpha} + 2\nu\frac{q^{\gamma N} - (-1)^N q^{\alpha N}}{q^\gamma + q^\alpha}\right) \\ & + \left(\frac{q^{\gamma(N+1)} - q^{\alpha(N+1)}}{q^\gamma - q^\alpha} + 2\nu\frac{q^{\gamma(N+1)} - (-1)^{(N+1)} q^{\alpha(N+1)}}{q^\gamma + q^\alpha}\right)\Bigg\}. \end{aligned} \tag{77}$$

In order to study the spectrum and the nonlinear frequency of this oscillator, it is convenient to use new parametrization

$$q = e^{\tau}, \qquad \alpha = \rho - \mu, \qquad \gamma = \rho + \mu. \tag{78}$$

We have

$$\begin{aligned} E_n = \frac{\hbar\omega_0}{2} \mathrm{e}^{\tau(\beta+\rho n)} \Bigg\{ & \frac{\sinh(\tau\mu(n+1))}{\sinh(\tau\mu)} + \mathrm{e}^{-\tau\rho} \frac{\sinh(\tau\mu n)}{\sinh(\tau\mu)} \\ & + 2\nu \frac{1-(-1)^n}{2} \left(\frac{\sinh(\tau\mu(n+1))}{\cosh(\tau\mu)} + \mathrm{e}^{-\tau\rho} \frac{\cosh(\tau\mu n)}{\cosh(\tau\mu)} \right) \\ & + 2\nu \frac{1+(-1)^n}{2} \left(\frac{\cosh(\tau\mu(n+1))}{\cosh(\tau\mu)} + \mathrm{e}^{-\tau\rho} \frac{\sinh(\tau\mu n)}{\cosh(\tau\mu)} \right) \Bigg\}, \end{aligned} \tag{79}$$

or

$$\begin{aligned} E_n = \frac{\hbar\omega_0}{2} \mathrm{e}^{\tau(\beta-\rho)} \Bigg\{ & \frac{1+\mathrm{e}^{\tau(\rho+\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho+\mu)n}}{\sinh(\tau\mu)} - \frac{1+\mathrm{e}^{\tau(\rho-\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho-\mu)n}}{\sinh(\tau\mu)} \\ & + 2\nu \frac{1-(-1)^n}{2} \left(\frac{1+\mathrm{e}^{\tau(\rho+\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho+\mu)n}}{\cosh(\tau\mu)} + \frac{1-\mathrm{e}^{\tau(\rho-\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho-\mu)n}}{\cosh(\tau\mu)} \right) \\ & + 2\nu \frac{1+(-1)^n}{2} \left(\frac{1+\mathrm{e}^{\tau(\rho+\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho+\mu)n}}{\cosh(\tau\mu)} - \frac{1-\mathrm{e}^{\tau(\rho-\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho-\mu)n}}{\cosh(\tau\mu)} \right) \Bigg\}. \end{aligned} \tag{80}$$

It follows that the number of the energy levels is infinite, and the eigenvalues E_n for $\tau \neq 0, n \to \infty$ depend on the parameters α, γ via the exponential factor $\mathrm{e}^{\tau(\rho-|\mu|)}$.

It is convenient to consider the eigenvalue E_n of H for n even and odd separately

$$\begin{aligned} E_n = \frac{\hbar\omega_0}{2} \mathrm{e}^{\tau(\beta-\rho)} \Bigg\{ & \frac{1+\mathrm{e}^{\tau(\rho+\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho+\mu)n}}{\sinh(\tau\mu)} - \frac{1+\mathrm{e}^{\tau(\rho-\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho-\mu)n}}{\sinh(\tau\mu)} \\ & + 2\nu \left(\frac{1+\mathrm{e}^{\tau(\rho+\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho+\mu)n}}{\cosh(\tau\mu)} + \frac{1-\mathrm{e}^{\tau(\rho-\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho-\mu)n}}{\cosh(\tau\mu)} \right) \Bigg\} \end{aligned} \tag{81}$$

for n odd, and

$$\begin{aligned} E_n = \frac{\hbar\omega_0}{2} \mathrm{e}^{\tau(\beta-\rho)} \Bigg\{ & \frac{1+\mathrm{e}^{\tau(\rho+\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho+\mu)n}}{\sinh(\tau\mu)} - \frac{1+\mathrm{e}^{\tau(\rho-\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho-\mu)n}}{\sinh(\tau\mu)} \\ & + 2\nu \left(\frac{1+\mathrm{e}^{\tau(\rho+\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho+\mu)n}}{\cosh(\tau\mu)} - \frac{1-\mathrm{e}^{\tau(\rho-\mu)}}{2} \frac{\mathrm{e}^{\tau(\rho-\mu)n}}{\cosh(\tau\mu)} \right) \Bigg\} \end{aligned} \tag{82}$$

for n even. From this the spectrum of this oscillator is not equidistant and the spacing is equal

$$E_{2n+1} - E_{2n} = \frac{\hbar\omega_0}{2} \frac{\sinh 2\mu\tau}{\sinh(\tau\mu)} \mathrm{e}^{\tau(\beta+\rho)} \mathrm{e}^{2\gamma\tau n}. \tag{83}$$

(h) Asymptotic behavior of the energy spectrum of Hamiltonian H.

According to the analysis given above, we have to give the estimation of the spectrum of Hamiltonian (80) for the parameter $-1 < 2\nu$, if $\tau(\gamma-\alpha) > 0$ and $-1 < 2\nu < \coth(\tau(\gamma-\alpha)/2)$ if $\tau(\gamma-\alpha) < 0$. Three cases must be distinguished: $\tau(\rho+|\mu|) > 0$, $\tau(\rho+|\mu|) < 0$, $\tau(\rho+|\mu|) = 0$.

In special case $\mu = 0$, $(\alpha = \gamma = \rho)$ we have

$$E_n = \frac{\hbar\omega_0}{2} \mathrm{e}^{\tau(\beta+\rho n)} [(n+\nu)(1+\mathrm{e}^{-\tau\rho}) + (-1)^n \nu (1-\mathrm{e}^{-\tau\rho}) + 1]. \tag{84}$$

If $(\tau(\rho+|\mu|) > 0$ and $-1 < 2\nu$, then from (80) it implies that the energy grows to infinity with increasing n.

If $\tau(\rho+|\mu|)=0$, then in the case $(\rho=\mu)$ we have

$$E_n=\frac{\hbar\omega_0}{2}\,\mathrm{e}^{\tau\beta}\,\mathrm{e}^{\beta-\rho}\left\{\frac{\mathrm{e}^{2\tau\rho n}-1}{\mathrm{e}^{\tau\mu}-\mathrm{e}^{-\tau\mu}}+\frac{\mathrm{e}^{2\tau\rho(n+1)}-1}{\mathrm{e}^{\tau\mu}-\mathrm{e}^{-\tau\mu}}+2\nu\frac{\mathrm{e}^{2\rho\tau n}-\mathrm{e}^{2\rho\tau(n+1)}}{\mathrm{e}^{\tau\mu}-\mathrm{e}^{-\tau\mu}}\right\}. \tag{85}$$

The same holds for $\rho+\mu=0$.

It follows that E_n monotonically increases to upper bounds that depends on parameters

$$E_{\max}=\frac{\hbar\omega_0}{2}\frac{\mathrm{e}^{\tau(\beta-\rho)}}{\sinh\tau|\mu|}.$$

If $\tau(\rho+|\mu|)<0$ and $-1<2\nu$ then the energies (81) and (82) at first increase and then go to zero as $n\to\infty$. In the particular case $\mu=0$ in (81) and (82) we have

$$E_n=\frac{\hbar\omega_0}{2}\,\mathrm{e}^{\tau(\beta+\rho n)}\{n(1+\mathrm{e}^{-\tau\rho})+2\nu+1\},$$

for n odd and

$$E_n=\frac{\hbar\omega_0}{2}\,\mathrm{e}^{\tau(\beta+\rho n)}\{n(1+\mathrm{e}^{-\tau\rho})+2\nu\,\mathrm{e}^{-\tau\rho}+1\}$$

for n even.

7. Conclusions

In this paper we have proposed the five-parameter $(q;\alpha,\beta,\gamma;\nu)$-deformation of the one-mode oscillator algebra. We have calculated the structure function of this deformation. The classifications of the irreducible representations are obtained.

We have extracted the deformed oscillator with the discrete spectrum of its 'quantized coordinate' and the eigenfunctions, expressed by means of the q-deformed (generalized) Hermitian I polynomials.

The asymptotic behavior of energy spectrum of the $(q;\alpha,\beta,\gamma;\nu)$-deformed Hamiltonian has been analyzed.

The nonlinear vector coherent states of this oscillator algebra for the special values of the deformation parameters have been constructed in [16].

Such algebras can be useful for the solution of the current problems in cosmology [40] and, as pointed out in [41], for noncommutative quantum field theory.

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